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# Remarks on the distributional Schwarzschild geometry

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This work is devoted to a mathematical analysis of the distributional Schwarzschild geometry. The Schwarzschild solution is extended to include the singularity; the energy momentum tensor becomes a $\delta$-distribution supported at $r = 0$. Using generalized distributional geometry in the sense of Colombeau's (special) construction the nonlinearities are treated in a mathematically rigorous way. Moreover, generalized function techniques are used as a tool to give a unified discussion of various approaches taken in the literature so far; in particular we comment on geometrical issues.

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## I. INTRODUCTION

Since the formulation of the first singularity theorems it is generally conceded that singular spacetimes are of fundamental importance in general relativity. Geometrically, a singularity is defined via the notion of (geodesic) incompleteness, a viewpoint which fits in the singularity theorems of Hawking and Penrose (see, e.g., [1], Chap. 8), forcing us to regard a singularity as some kind of singular boundary point of spacetime. Recently, as an alternative, it has been suggested to describe (mild) singularities as internal points, where the field equations are satisfied in a weak (probably distributional) sense (cf. [2]). General relativity as a physical theory is governed by particular physical equations; the focus of interest is the breakdown of physics which need not coincide with the breakdown of geometry.

In the context of conical spacetimes algebras of generalized functions [3, 4] have been used to overcome the problem of simultaneously dealing with singular (i.e., distributional) metrics and the nonlinearities of general relativity [5, 6, 7]. These techniques allow to assign to the cone metric a singular energy momentum tensor supported on a submanifold of codimension two, which, by a result of Geroch and Traschen [8], is not possible within classical (i.e., linear) distribution theory.

The main focus of this work is a (nonlinear) distributional description of the Schwarzschild spacetime. Although the nature of the Schwarzschild singularity is much "worse" than the quasi-regular conical singularity, there are several distributional treatments in the literature ([9, 10, 11, 12, 13]), mainly motivated by the following considerations: the physical interpretation of the Schwarzschild metric is clear as long as we consider it merely as an exterior (vacuum) solution of an extended (sufficiently large) massive spherically symmetric body. Together with the interior solution it describes the entire spacetime. The concept of point particles—well understood in the context of linear field theories—suggests a mathematical idealization of the underlying physics: one would like to view the Schwarzschild solution as defined on the entire spacetime and regard it as generated by a point mass located at the origin and acting as the gravitational source. This of course amounts to the question of whether one can reasonably ascribe distributional curvature quantities to the Schwarzschild singularity at the origin.

The emphasis of the present work lies on mathematical rigor. We derive the "physically expected" result for the distributional energy momentum tensor of the Schwarzschild geometry, i.e., $T^0_0 = 8\pi m \delta^{(3)}(\vec{x})$, in a conceptually

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satisfactory way. Additionally, we set up a unified language to comment on the respective merits of some of the approaches taken so far. In particular, we discuss questions of differentiable structure as well as smoothness and degeneracy problems of the regularized metrics, and present possible refinements and workarounds. These aims are accomplished using the framework of nonlinear generalized functions (Colombeau algebras) [3, 4] and, in particular, the geometric approach taken in [14, 15].

The paper is organized in the following way: in section II we discuss the conceptual as well as the mathematical prerequisites. In particular we comment on geometrical matters (differentiable structure, coordinate invariance) and recall the basic facts of nonlinear distributional geometry in the context of algebras of generalized functions. Moreover, we derive sensible regularizations of the singular functions to be used throughout the paper. Section III is devoted to a first approach to the problem; a detailed discussion follows in section IV: we comment on problems and obstacles associated with the direct approach. Finally, in section V, we present a new conceptually satisfactory method to derive the main result. Overly technical calculations are shifted to various appendices. In the final section VI we investigate the horizon and describe its distributional curvature. Using nonlinear distributional geometry and generalized functions it seems possible to show that the horizon singularity is only a coordinate singularity *without leaving Schwarzschild coordinates*.

## II. PREREQUISITES

To begin with, let us have a look at the conceptually much simpler problem of point charges in Maxwell's theory and consider the Coulomb solution $\frac{1}{r}$ of an extended spherically symmetric body. In an idealized picture the charged body is reduced to a point charge, and this way of looking at the problem has proven to be very fruitful, mainly due to the following two reasons: first, the function $\frac{1}{r} \in \mathcal{C}^2(\mathbb{R}^3 \backslash \{0\})$, since also in $L^1_{loc}(\mathbb{R}^3)$, naturally gives rise to a distribution on $\mathbb{R}^3$. Reinserting this distributional potential into the field equation we obtain $\Delta \frac{1}{r} = -4\pi\delta$, which has the clear physical interpretation as the charge density of a point charge. Second, also in accordance with physical intuition, the situation may be interpreted in terms of the following sensible regularization scenario: consider a regularization of the "singular" potential by any sequence of (say smooth) functions converging weakly to $\frac{1}{r}$. Then, by virtue of linearity of the field equation, distribution theory guarantees that the corresponding sequence of charge densities will converge weakly to $-4\pi\delta$, i.e., the density of the point charge.

The general relativistic case is much more involved. Consider the Schwarzschild metric inside the horizon: extending the spacetime to $r = 0$ we are confronted with several distinct problems. First—according to the standard picture of general relativity—no manifold structure is given at the singularity $r = 0$, since the field equations are meaningless there within the smooth category. In addition, the differentiable structure of the extended manifold cannot be uniquely determined from the differentiable structure of the original spacetime. This problem is dealt with in the relevant literature by fixing some differentiable structure by hand, most often the one induced by Cartesians associated to Schwarzschild coordinates.

In analogy to the Maxwell case, we want to regard the metric as a distribution on the whole extended spacetime. Now, the second conceptual problem is due to the inherently nonlinear nature of general relativity: no distributional meaning can be given to the field equations, since it is not possible to calculate the curvature from a distributional metric. In the literature, this obstacle is circumvented by using various—more or less—ad-hoc regularization approaches in order to calculate a regularized Ricci tensor within the smooth category. Eventually, its distributional limit is computed and—via the field equations—a distributional energy momentum tensor is obtained. This tensor may then be interpreted as distributional source of the Schwarzschild geometry [9, 10, 11, 12, 13]. However, using ad-hoc regularizations we are confronted with the problem of regularization independence of the results which may not be suitably addressed within this setting.

In this work, while arguing form a related point of view, we are going to use a different apparatus to deal with the nonlinearities of GR: the theory of algebras of generalized functions gives us the additional flexibility and power of a rigorous mathematical framework in which distributions may undergo nonlinear operations. In particular, following the procedure of [5], we will first model the distributionally extended Schwarzschild metric by a generalized metric obtained by a suitable (and general) regularization procedure. Then, after entering the generalized framework (cf. [15]) we may calculate all the relevant curvature quantities from the generalized metric and subject it to the field equations. Note that within the generalized setting the field equations possess a well defined meaning. Finally, we may descend to the distributional level for the purpose of interpretation using the concept of association (see below).

Note that, in general, a regularization procedure depends on the coordinate system it is performed in (for a diffeomorphism invariant notion of regularization using paths of mollifiers see [16, 17]). However, since we had to fix the differentiable structure beforehand this is no further restriction. Actually, geometric considerations play an important role: as shown below, a brute force regularization attempt does not lead to a sensible description of the

problem at hand. Indeed, we shall see that a satisfactory treatment of the distributional Schwarzschild spacetime has to use the Kerr-Schild form of the Schwarzschild metric (which fixes both the differentiable structure and the coordinates); moreover, it must be retained during the whole regularization process. Note that this is in accordance with physical intuition since in the Kerr-Schild form the radial coordinate retains its spacial character near the singularity which of course is not the case in Schwarzschild coordinates.

In the remainder of this section we are going to introduce some mathematical prerequisites. First, we are going to shortly recall generalized tensor analysis and generalized curvature (in Colombeau's so-called special setting). For all further details we refer the reader to [14, 15]. Second, we explicitly calculate the regularization of the relevant components of the metric tensor to be used throughout the paper.

**Nonlinear distributional geometry**

The basic idea of Colombeau's theory of generalized functions [3, 4] is regularization by sequences (nets) of smooth functions and the use of asymptotic estimates in terms of a regularization parameter $\varepsilon$. Let $(u_\varepsilon)_{\varepsilon\in(0,1]}$ with $u_\varepsilon \in \mathcal{C}^\infty(M)$ for all $\varepsilon$ ($M$ a separable, smooth orientable Hausdorff manifold of dimension $n$). The algebra of generalized functions on $M$ is defined as the quotient $\mathcal{G}(M) := \mathcal{E}_M(M)/\mathcal{N}(M)$ of the space $\mathcal{E}_M(M)$ of sequences of moderate growth modulo the space $\mathcal{N}(M)$ of negligible sequences. More precisely the notions of moderateness resp. negligibility are defined by the following asymptotic estimates ($\mathfrak{X}(M)$ denoting the space of smooth vector fields on $M$)

$$\begin{aligned}
\mathcal{E}_M(M) &:= \{(u_\varepsilon)_\varepsilon : \ \forall K \subset\subset M, \forall k \in \mathbb{N}_0 \, \exists N \in \mathbb{N} \\
&\qquad \forall \xi_1, \ldots, \xi_k \in \mathfrak{X}(M) : \ \sup_{p\in K} |L_{\xi_1} \ldots L_{\xi_k} u_\varepsilon(p)| = O(\varepsilon^{-N})\} \\
\mathcal{N}(M) &:= \{(u_\varepsilon)_\varepsilon : \ \forall K \subset\subset M, \forall k, q \in \mathbb{N}_0 \\
&\qquad \forall \xi_1, \ldots, \xi_k \in \mathfrak{X}(M) : \ \sup_{p\in K} |L_{\xi_1} \ldots L_{\xi_k} u_\varepsilon(p)| = O(\varepsilon^{q}))\}
\end{aligned}$$

Elements of $\mathcal{G}(M)$ are denoted by $u = \mathrm{cl}[(u_\varepsilon)_\varepsilon] = (u_\varepsilon)_\varepsilon + \mathcal{N}(M)$. With componentwise operations $\mathcal{G}(M)$ is a fine sheaf of differential algebras with respect to the Lie derivative defined by $L_\xi u := \mathrm{cl}[(L_\xi u_\varepsilon)_\varepsilon]$. The spaces of moderate resp. negligible sequences and hence the algebra itself may be characterized locally, i.e., $u \in \mathcal{G}(M)$ iff $u \circ \psi_\alpha \in \mathcal{G}(\psi_\alpha(V_\alpha))$ for all charts $(V_\alpha, \psi_\alpha)$, where on the open set $\psi_\alpha(V_\alpha) \subset \mathbb{R}^n$ in the respective estimates Lie derivatives are replaced by partial derivatives. Smooth functions are embedded into $\mathcal{G}$ simply by the "constant" embedding $\sigma$, i.e., $\sigma(f) := \mathrm{cl}[(f)_\varepsilon]$, hence $\mathcal{C}^\infty(M)$ is a faithful subalgebra of $\mathcal{G}(M)$. On open sets of $\mathbb{R}^n$ compactly supported distributions are embedded into $\mathcal{G}$ via convolution with a mollifier $\rho \in \mathcal{S}(\mathbb{R}^n)$ with unit integral satisfying $\int \rho(x)x^\alpha dx = 0$ for all $|\alpha| \geq 1$; more precisely setting $\rho_\varepsilon(x) = (1/\varepsilon^n)\rho(x/\varepsilon)$ we have $\iota(w) := \mathrm{cl}[(w * \rho_\varepsilon)_\varepsilon]$. In case $\mathrm{supp}(w)$ is not compact one uses a sheaf-theoretical construction. However, in the special case of the functions to be treated in the context of the Schwarzschild metric this will not be necessary (see below). From the explicit formula it is clear that (on open subsets of Euclidean space) embedding commutes with partial differentiation. On a general manifold, however, there is no canonical embedding of $\mathcal{D}'$ available; a suitable replacement (cf. [14]) is provided by physically motivated modeling and the use of the notion of association (see below). Inserting $p \in M$ into $u \in \mathcal{G}(M)$ yields a well defined element of the ring of constants (also called generalized numbers) $\mathcal{K}$ (corresponding to $\mathbb{K} = \mathbb{R}$ resp. $\mathbb{C}$), defined as the set of moderate nets of numbers ($(r_\varepsilon)_\varepsilon \in \mathbb{K}^{(0,1]}$ with $|r_\varepsilon| = O(\varepsilon^{-N})$ for some $N$) modulo negligible nets ($|r_\varepsilon| = O(\varepsilon^m)$ for each $m$). Finally, generalized functions on $M$ are characterized by their generalized point values, i.e., by their values on points in $\tilde{M}_c$, the space of equivalence classes of compactly supported nets $(p_\varepsilon)_\varepsilon \in M^{(0,1]}$ with respect to the relation $p_\varepsilon \sim p'_\varepsilon :\Leftrightarrow d_h(p_\varepsilon, p'_\varepsilon) = O(\varepsilon^m)$ for all $m$, where $d_h$ denotes the distance on $M$ induced by any Riemannian metric.

The $\mathcal{G}(M)$-module of generalized sections in vector bundles—especially the space of generalized tensor fields $\mathcal{G}^r_s(M)$—is defined along the same lines using analogous asymptotic estimates with respect to the norm induced by any Riemannian metric on the respective fibers. However, it is more convenient to use the following algebraic description of generalized tensor fields

$$\mathcal{G}^r_s(M) = \mathcal{G}(M) \otimes \mathcal{T}^r_s(M)\,, \tag{1}$$

where $\mathcal{T}^r_s(M)$ denotes the space of smooth tensor fields and the tensor product is taken over the module $\mathcal{C}^\infty(M)$. Hence generalized tensor fields are just given by classical ones with generalized coefficient functions. Many concepts of classical tensor analysis carry over to the generalized setting [14], in particular Lie derivatives with respect to both classical and generalized vector fields, Lie brackets, exterior algebra, etc. Moreover, generalized tensor fields may also be viewed as $\mathcal{G}(M)$-multilinear maps taking generalized vector and covector fields to generalized functions, i.e., as

$\mathcal{G}(M)$-modules we have

$$\mathcal{G}^r_s(M) \cong L_{\mathcal{G}(M)}(\mathcal{G}^0_1(M)^r, \mathcal{G}^1_0(M)^s; \mathcal{G}(M)).$$

In particular a generalized metric is defined to be a symmetric, generalized $(0,2)$-tensor field $g_{ab} = \mathrm{cl}[(g_{ab\,\varepsilon})_\varepsilon]$ (with its index independent of $\varepsilon$ and) whose determinant $\det(g_{ab})$ is invertible in $\mathcal{G}(M)$. The latter condition is equivalent to the following notion called strictly nonzero on compact sets: for any representative $\det(g_{ab\,\varepsilon})_\varepsilon$ of $\det(g_{ab})$ we have $\forall K \subset\subset M\ \exists m \in \mathbb{N}:\ \inf_{p\in K} |\det(g_{ab\,\varepsilon})| \geq \varepsilon^m$ for all $\varepsilon$ small enough. This notion captures the intuitive idea of a generalized metric to be a sequence of classical metrics approaching a singular limit in the following sense: $g_{ab}$ is a generalized metric iff (on every relatively compact open subset $V$ of $M$) there exists a representative $(g_{ab\,\varepsilon})_\varepsilon$ of $g_{ab}$ such that for fixed $\varepsilon$ (small enough) $g_{ab\,\varepsilon}$ (resp. $g_{ab\,\varepsilon}|_V$) is a classical pseudo-Riemannian metric and $\det(g_{ab})$ is invertible in the algebra of generalized functions. A generalized metric induces a $\mathcal{G}(M)$-linear isomorphism from $\mathcal{G}^1_0(M)$ to $\mathcal{G}^0_1(M)$ and the inverse metric $g^{ab} := \mathrm{cl}[(g^{-1}_{ab\,\varepsilon})_\varepsilon]$ is a well defined element of $\mathcal{G}^2_0(M)$ (i.e., independent of the representative $(g_{ab\,\varepsilon})_\varepsilon$). Also the generalized Levi-Civita connection as well as the generalized Riemann-, Ricci- and Einstein tensor of a generalized metric are defined simply by the usual coordinate formulae on the level of representatives.

Finally, the setting introduced above displays maximal consistency (in the light of L. Schwartz impossibilty result [18]) with respect to smooth resp. distributional geometry most conveniently formalized in terms of the notion of association. A generalized function $u \in \mathcal{G}(M)$ is called associated to zero, $u \approx 0$, if one (hence any) representative $(u_\varepsilon)_\varepsilon$ converges to zero weakly. (In a sloppy fashion we shall often write $u_\varepsilon \approx 0$.) The equivalence relation $u \approx v :\Leftrightarrow u-v \approx 0$ gives rise to a linear quotient of $\mathcal{G}$ that extends distributional equality. Moreover we call a distribution $w \in \mathcal{D}'(M)$ the distributional shadow or macroscopic aspect of $u$ and write $u \approx w$ if for all compactly supported $n$-forms $\nu$ and one (hence any) representative $(u_\varepsilon)_\varepsilon$

$$\lim_{\varepsilon\to 0} \int\limits_M u_\varepsilon \nu = \langle w, \nu \rangle,$$

where $\langle\,,\,\rangle$ denotes the distributional action. By (1) the concept of association extends to generalized tensor fields in a natural way.

### Regularizations of the singular functions occurring in the Schwarzschild metric

The two most important singular functions we will work with throughout this paper (namely the singular components of the Schwarzschild metric) are $\frac{1}{r}$ and $\frac{1}{r-c}$ ($r = \|\vec{x}\|$ ; $c$ a positive constant). Since $\frac{1}{r} \in L^1_{loc}(\mathbb{R}^3)$, it gives rise to the regular distribution $\frac{1}{r} \in \mathcal{D}'(\mathbb{R}^3)$. By convolution with a mollifier $\rho$ (adapted to the symmetry of the spacetime, thus chosen radially symmetric) we embed it into the Colombeau algebra $\mathcal{G}(\mathbb{R}^3)$

$$\frac{1}{r} \quad \stackrel{\iota}{\longrightarrow} \quad \iota(\frac{1}{r}) = (\frac{1}{r} * \rho_\varepsilon) =: (\frac{1}{r})_\varepsilon\,. \tag{2}$$

Using radial symmetry of the convoluted function and inserting $\rho_\varepsilon(r) = \frac{1}{\varepsilon^3}\rho(\frac{r}{\varepsilon})$ we obtain

$$(\frac{1}{r})_\varepsilon = \frac{4\pi}{r} \int\limits_0^{r/\varepsilon} dt\, t^2\, \rho(t) + \frac{4\pi}{\varepsilon} \int\limits_{r/\varepsilon}^{\infty} dt\, t\, \rho(t)\,. \tag{3}$$

It is easy to confirm that $(\frac{1}{r})_\varepsilon = \sigma(\frac{1}{r})_\varepsilon = \frac{1}{r}$ on $\mathbb{R}^3\backslash\{0\}$, and at the origin we have $(\frac{1}{r})_\varepsilon\Big|_{r=0} = \frac{4\pi}{\varepsilon}\int_0^\infty dt\, t\rho(t)\,.$

In contrast to $\frac{1}{r}$, the function $\frac{1}{r-c}$ is not in $L^1_{loc}(\mathbb{R}^3)$. A canonical regularization (in the sense of Gelfand-Shilov [19]) is the principal value $\mathrm{v}p(\frac{1}{r-c}) \in \mathcal{D}'(\mathbb{R}^3)$ which can be embedded into $\mathcal{G}(\mathbb{R}^3)$.

$$\frac{1}{r-c} \quad \rightarrow \quad \mathrm{v}p(\frac{1}{r-c}) \in \mathcal{D}'(\mathbb{R}^3) \quad \stackrel{\iota}{\longrightarrow} \quad \iota(\mathrm{v}p(\frac{1}{r-c})) =: (\mathrm{v}p(\frac{1}{r-c}))_\varepsilon\,. \tag{4}$$

Making use of $\mathrm{v}p(\frac{1}{r-c}) = \frac{\partial}{\partial r}\log|r-c|$ we obtain $\iota(\mathrm{v}p(\frac{1}{r-c}))(x) =$

$$(1 + r\frac{\partial}{\partial r}) \int d^3y \frac{1}{|x-y|} \log|\,|x-y|\, - c|\, \rho_\varepsilon(y) - \frac{\partial}{\partial x^i} \int d^3y\, y^i \frac{1}{|x-y|} \log|\,|x-y|\, - c|\, \rho_\varepsilon(y)$$

and finally for $v \geq c$

$$\iota(\mathrm{v}p(\frac{1}{r-c}))(x) \;=\; \frac{4\pi}{r-c}\int\limits_0^{r-c} ds\rho_\varepsilon(s)s^2 + \frac{4\pi}{r}\int\limits_{r-c}^{\infty} ds\rho_\varepsilon(s)s^2 + \\ + \frac{4\pi}{r-c}\frac{c}{r}\int\limits_0^{r-c} ds\rho_\varepsilon(s)s^2\sum_{l=1}^{\infty}\frac{1}{2l+1}(\frac{s}{r-c})^{2l} + \\ + \frac{4\pi}{r-c}\frac{c}{r}\int\limits_{r-c}^{\infty} ds\rho_\varepsilon(s)(r-c)^2\sum_{l=0}^{\infty}\frac{1}{2l+1}(\frac{r-c}{s})^{2l}. \tag{5}$$

For $0 < r \leq c$ the roles of $r$ and $c$ are interchanged and at the origin we obtain $\mathrm{v}p(\frac{1}{r-c})_\varepsilon\Big|_{r=0} = -\frac{1}{c} + O(\varepsilon)$.

## III. A FIRST APPROACH TO THE PROBLEM

In this section we present a first approach to the "Schwarzschild point mass problem", thereby essentially following earlier treatments in the literature ([9, 11, 12, 13]). However, we are going to use the language of nonlinear distributional geometry introduced above in order to obtain a unified view, which will enable us to carry out a detailed analysis of the previous approaches in the next section.

In the usual Schwarzschild coordinates $(t, r > 0, \theta, \phi)$ the metric takes the form

$$ds^2 = h(r)\,dt^2 - h(r)^{-1}\,dr^2 + r^2\,d\Omega^2 \text{ with } h(r) = -1 + \frac{2m}{r}. \tag{6}$$

Following the above discussion we consider the singular metric coefficient $h(r)$ as an element of $L^1_{loc}(\mathbb{R}^3) \subseteq \mathcal{D}'(\mathbb{R}^3)$ and embed it into $\mathcal{G}(\mathbb{R}^3)$ by convolution with a mollifier. Note that, accordingly, we have fixed the differentiable structure of the manifold: the Cartesian coordinates associated with the spherical Schwarzschild coordinates in (6) are extended through the origin. We have

$$h(r) = -1 + \frac{2m}{r} \quad \xrightarrow{\iota} \quad \iota(h(r)) = h_\varepsilon(r) = -1 + 2m(\frac{1}{r})_\varepsilon \in \mathcal{G}(\mathbb{R}^3)\,, \tag{7}$$

where $(\frac{1}{r})_\varepsilon$ is given by (3). Inserting (7) into (6) we obtain a generalized object modeling the singular Schwarzschild metric, i.e.,

$$ds^2_\epsilon = h_\epsilon(r)\,dt^2 - h_\epsilon(r)^{-1}\,dr^2 + r^2\,d\Omega^2\,. \tag{8}$$

The generalized Ricci tensor may now be calculated componentwise using the classical formulae

$$(R^0{}_0)_\varepsilon = (R^1{}_1)_\varepsilon \;=\; \frac{1}{2}\left(h''_\varepsilon + \frac{2}{r}h'_\varepsilon\right) = \frac{1}{2}\,\Delta h_\varepsilon \tag{9}$$

$$(R^2{}_2)_\varepsilon = (R^3{}_3)_\varepsilon \;=\; \frac{h'_\varepsilon}{r} + \frac{1+h_\varepsilon}{r^2}\,. \tag{10}$$

Due to the linear structure of $R^0_0$ it is evident that it is associated to a constant times the $\delta$-distribution, i.e.,

$$(R^0{}_0)_\varepsilon = \frac{1}{2}\,\Delta h_\varepsilon = m\,\Delta(\frac{1}{r})_\varepsilon \;\to\; -4\pi m\delta \quad (\varepsilon \to 0). \tag{11}$$

Investigating the weak limit of the angular components of the Ricci tensor (using the abbreviation $\tilde{\Phi}(r) = \int \sin\theta d\theta d\phi\,\Phi(\vec{x})$) we get (cf. appendix A)

$$\int (R^2{}_2)_\varepsilon \Phi\, d^3x \;=\; \int (rh'_\varepsilon + 1 + h_\varepsilon)\tilde{\Phi}(r)dr = \\ \stackrel{(3)}{=}\; 8\pi m \int \frac{1}{\varepsilon}\,[\int_{r/\varepsilon}^{\infty} t\rho(t)dt]\,\tilde{\Phi}(r)dr = 8\pi m\int dx\,\tilde{\Phi}(\varepsilon x)\int_x^\infty t\rho(t)dt \\ \to\; 32\pi^2 m\,\Phi(0)\int\limits_0^\infty dx\int_x^\infty t\rho(t)dt \;\stackrel{(A3)}{=}\; 8\pi m\langle\delta|\Phi\rangle \qquad (\varepsilon \to 0).$$

Hence, the Ricci tensor and the curvature scalar $R$ are of $\delta$-type, i.e.,

$$R^0_{\ 0} \,=\, R^1_{\ 1} \approx -4\pi m\,\delta \qquad R^2_{\ 2} = R^3_{\ 3} \approx 8\pi m\,\delta \qquad R \approx \pi m\,\delta. \tag{12}$$

Equations (12) are obviously given in spherical coordinates. Strictly speaking this is not sensible, because the basis fields $\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \theta}\}$ are not globally defined. Representing distributions concentrated at the origin requires a basis regular at the origin. Transforming the results for $(R^i_{\ j})_\varepsilon$ (i.e., (9) and (10)) into Cartesian coordinates associated with the spherical ones (i.e., $\{r,\theta,\phi\} \leftrightarrow \{x^i\}$) we obtain, e.g., for the Einstein tensor

$$G^i_{\ j} \;\approx\; -8\pi m\,\delta\,\delta^i_0 \delta^0_j. \tag{13}$$

Note that the use of the particular regularization (7) is not essential here. We could have replaced (7) by any other smooth ad-hoc regularization of $h(r)$, as has been done, e.g., in [11], by setting $h_\varepsilon(r) = -1 + \frac{2m}{\sqrt{r^2+\varepsilon^2}}$. Indeed, we can show that the results (12) hold for all regularizations, i.e., for all sequences of the form $h_\varepsilon(r) = -1 + 2m s_\varepsilon(r) \to h(r)$ (i.e., $\forall\, s_\varepsilon$ smooth, $s_\varepsilon \to \frac{1}{r}$ in $\mathcal{D}'$). For the $(0,0)$- and $(1,1)$-components of the Ricci tensor the result follows from the special form of (9). For the angular components (cf. (10)) we write

$$2m \int\limits_0^\infty r^2 (\frac{s'_\varepsilon(r)}{r} + \frac{s_\varepsilon(r)}{r^2})\,\tilde{\Phi}(r)\,dr = -2m \int\limits_0^\infty dr\, r^2 s_\varepsilon\,\frac{1}{r}\,\tilde{\Phi}'(r) \;\to\; 8\pi m \Phi(0), \tag{14}$$

where in the last step we used the fact that $\frac{1}{r}\,\tilde{\Phi}' \in \mathcal{D}(\mathbb{R})$.

## IV. COMMENTS AND PROBLEMS

In order to be able to calculate the curvature from the metric we must keep the regularization $h_\varepsilon(r)$ smooth on the entire spacetime. This fact—although somewhat hidden because we worked with spherical coordinates—is essential from the conceptual point of view. In fact, choosing a regularization $h_\varepsilon(r)$ which is smooth only on $\mathbb{R}^3 \backslash \{0\}$ is not sufficient to derive the result as is explicitly shown by the following counterexample. Set $h_\varepsilon = -1 + 2m s_\varepsilon$ and $s_\varepsilon = \frac{1}{r}\, o_\varepsilon$ (with $o_\varepsilon \to 1$ weakly) consisting of regular distributions, so that $s_\varepsilon \in L^1_{loc}(\mathbb{R}^3)$ with $s_\varepsilon \to \frac{1}{r} \in \mathcal{D}'(\mathbb{R}^3)$. Moreover, we may require $o_\varepsilon(r)$ to be smooth on $\mathbb{R}^3 \backslash \{0\}$. Summing over (9) and (10) we get $R_\varepsilon = 2m(\frac{1}{r}\, o''_\varepsilon + \frac{2}{r^2}\, o'_\varepsilon)$. Choosing $o_\varepsilon(r) = (1 + c\,[r^\varepsilon - 1])$ we obtain for $R_\varepsilon$ different weak limits as the constant $c$ varies, i.e.,

$$R_\varepsilon \;\to\; 8\pi m c\,\delta. \tag{15}$$

For $o_\varepsilon = r^{-\varepsilon}$ the situation is even worse. Although $o_\varepsilon \in L^1_{loc}\,\forall\,\varepsilon$ and $o_\varepsilon \to 1 \in \mathcal{D}'$ as $\varepsilon \to 0$, $R_\varepsilon$ does not converge weakly, so that we obtain no distributional result whatsoever.

Nonetheless, similar non-smooth regularizations have been considered in the literature. In these cases the desired result (12) can only be produced by means of a clever choice of explicit formulae; in particular, $o_\varepsilon = r^\varepsilon$ in [13] and $o_\varepsilon(r) = \Theta(r - \varepsilon)$ in [12]. The authors of [9] have shown that the result (12) may be reproduced as long as $o_\varepsilon|_{r=0} = 0$. However, the conceptual problem remains untouched: $R_\varepsilon[h_\varepsilon]$ can only be derived for smooth regularizations $h_\varepsilon$; distributions cannot be used as an input for nonlinear operations.

Prior to a more detailed investigation of the choice of regularization, we briefly comment on two more attempts in the literature. In [12] a regularization of the metric using thin shell solutions is investigated. The limit $(\varepsilon \to 0)$ corresponds to a shrinking of the shell. However, the shells can only be placed outside the horizon (of a black hole with identical mass). This implies that a shrinking of the shell must be coupled to a decrease in mass: $m$ converges to zero in the limiting process, so the obtained results should either be considered trivial ($R \sim m\delta\,|_{m=0} = 0$) or be rejected completely.

In [11] the authors claim to have found different results for the curvature quantities by regularizing the Schwarzschild metric in a different coordinate system. They study the interrelations of regularizations and coordinate transformations for this particular problem. However, some details are not overly convincing. If we choose a new radial coordinate $\tilde{r}$ such that $r = \Lambda\tilde{r} + a$ with $a = 2m$, then $\tilde{r} = 0$ does not describe the Schwarzschild singularity. Instead, $\tilde{r} = 0$ corresponds to the coordinate singularity at the horizon $r = 2m$, but shrunk to one point. Obviously, we should not compare the outcome of these considerations with our former results.

We now begin with an in-depth analysis of certain aspects of the regularization procedure commencing with the issue of componentwise regularization and invertibility of the regularized metric. According to (1) in section II, regularizing

a tensor such as the Schwarzschild metric (6) comes up to regularizing each distribution-valued component separately. Following this rule we obtain a regularized metric slightly different from (8), namely

$$ds_\varepsilon^2 = h_\varepsilon(r)\, dt^2 - (h^{-1})_\varepsilon(r)\, dr^2 + r^2\, d\Omega^2\,. \tag{16}$$

Since $\mathrm{cl}[h_\varepsilon]\mathrm{cl}[(h^{-1})_\varepsilon] \neq 1 \in \mathcal{G}$, the determinant of the regularized metric (16) is no longer identically one. (This, in fact, does not come as a surprise; cf. Schwartz' impossibility result [18].) However, the product *is* preserved in the sense of association, i.e., $h_\varepsilon(h^{-1})_\varepsilon \approx 1$. Analogous issues arise from the inverse metric: embedding also $g^{-1}$ componentwise into $\mathcal{G}$ we obtain regularized objects, $g_\varepsilon$ and $(g^{-1})_\varepsilon$, which are only inverse to each other in the sense of association. Taking a different viewpoint, however, it is comparatively easy to avoid these issues: on the classical level the Schwarzschild geometry is uniquely determined by the set of variables $\{g_{tt}, g_{rr}, g_{\theta\theta}, g_{\varphi\varphi}\}$, or, e.g., equivalently by $\{g_{tt}, g_{\theta\theta}, g_{\varphi\varphi}, \det g\}$. Embedding the second set of variables into $\mathcal{G}$ leads directly to the regularization (8) used above; no invertibility problems arise at all since $\det g_\varepsilon$ is forced to equal one.

Finally we return to discussing the problem of smoothness of the regularized metric from a different, more geometrical point of view. We regard this problem to be so essential that in the next section we propose an approach entirely different form the one taken so far.

In fact, the regularizations used so far (as all the other regularizations in the relevant literature) do *not* provide a *smooth* regularized metric tensor. This fact is hidden again by the use of spherical coordinates. In Cartesian coordinates pertaining to $(r, \theta, \phi)$—which we used to fix the differentiable structure of the extended manifold at $r = 0$—however, it can be explicitly seen from the form of the metric

$$ds^2 = h(r)\, dt^2 + d\vec{x}^2 - (1 + h(r)^{-1})\frac{x_i x_j}{r^2}\, dx^i dx^j\,. \tag{17}$$

In order to obtain a smooth regularization it is not sufficient to merely regularize $h(r)$. In fact, we must embed the singular coefficient $(1 + h(r)^{-1})\frac{x_i x_j}{r^2}$ as a whole into $\mathcal{G}$. Apart from technical difficulties this approach should provide a smooth regularized metric $ds_\varepsilon$. However, we have reached an impasse: the regularized metric will not be invertible at some distinct value $r_0$ of the radial coordinate, where $r_0 \to 0$ $(\varepsilon \to 0)$. This will be shown in the remainder of this section.

As shown in appendix B, the regularization of (17) takes the form

$$ds_\varepsilon^2 = h_\varepsilon(r)\, dt^2 + (1 - a_\varepsilon(r))\, dr^2 + (1 - b_\varepsilon(r))\, r^2 d\Omega^2, \tag{18}$$

with $a_\varepsilon(0) \to \frac{1}{3}$ $(\varepsilon \to 0)$. This implies that the $rr$-component of the regularized metric (18) is positive at $r = 0$ (at least for small $\varepsilon$), because $(g_{rr})_\varepsilon(0) = (1 - a_\varepsilon(0)) \to \frac{2}{3}$ $(\varepsilon \to 0)$. On the other hand, $(1 - a_\varepsilon)$ approximates $-h^{-1}$, i.e., $(g_{rr})_\varepsilon(r \neq 0) \to -\frac{r}{2m-r} < 0 \quad (\varepsilon \to 0)$. So we conclude that at some value $r_0$ of $r$ the smooth function $(g_{rr})_\varepsilon(r)$ must have a zero at least for small $\varepsilon$. (Interestingly enough, this is not the case for negative masses since $-\frac{r}{2m-r}$ is positive then). In other words, this means that the regularization of the metric (17) degenerates at some radius $r_0$. Evidently, $r_0 \to 0$ as $\varepsilon \to 0$.

Note that the occurrence of this radius of degeneracy is neither due to the fact that we choose the particular regularization (18), nor is it possible to avoid it by giving up spherical symmetry. To see this in some more detail consider the spatial part of (17) (set $\tilde{h}(r) := h^{-1}(r)$) and consider a certain class of regularizations

$$ds_\varepsilon^2 = d\vec{x}^2 - (1 + \tilde{h}_\varepsilon(r))\, \frac{x_i x_j}{r^2}\, dx^i dx^j, \tag{19}$$

where $\tilde{h}_\varepsilon(r)$ denotes an arbitrary regularization of $\tilde{h}(r)$. However, for $ds_\varepsilon^2$ to become smooth, we must require that $\tilde{h}_\varepsilon(r)$ be $-1 + O(r^2)$ for $(r \to 0)$. Now, an arbitrary regularization of $ds^2$ not necessarily respecting spherical symmetry is obtained by adding zero-sequences to (19).

$$ds_\varepsilon^2 = d\vec{x}^2 - (1 + \tilde{h}_\varepsilon(r))\, \frac{x_i x_j}{r^2}\, dx^i dx^j + (a_{ij})_\varepsilon(\vec{x})\, dx^i dx^j. \tag{20}$$

For special cases of $(a_{ij})_\varepsilon$ it is easy to show that (20) is degenerate. Choose, e.g., $(a_{ij})_\varepsilon$ such that only $(a_{12})_\varepsilon =: b_\varepsilon(\vec{x})$ is non-vanishing. The determinant of (20) at $\vec{x} = (0, 0, r)$ is equal to $-\tilde{h}_\varepsilon(r)(1 - b_\varepsilon(\vec{x})^2)$. As $-\tilde{h}_\varepsilon(0) = 1$ and $-\tilde{h}_\varepsilon(r) < 0$ for small $\varepsilon$ and finite $r$, there exists a radius $r_\varepsilon$ (with $r_\varepsilon \to 0$ for $\varepsilon \to 0$), such that $\det(g_\varepsilon) = 0$. Again, we observe degeneracy.

## V. THE KERR-SCHILD APPROACH

To begin with let us summarize what we have done so far: we considered regularizations of the Schwarzschild metric (using the language of algebras of generalized functions) to calculate the (distributional) curvature at the singularity. The regularizations used were essentially based on Cartesian coordinates associated with the spherical Schwarzschild coordinates. However, it turned out that all these regularizations were either non-smooth or not invertible. *Smoothness and invertibility mutually exclude each other* in this context. Hence, we are going to take another more geometrical view-point in this section. The main idea—following [10, 20]—is to use the Kerr-Schild form of the Schwarzschild metric. Retaining this preferred structure also during the whole regularization process will enable us to derive the physically desired result in a rigorous manner.

A metric belongs to the so-called Kerr-Schild class [21] if it can be written as

$$g_{ij} = \eta_{ij} + f k_i k_j \qquad \text{with} \quad k^i k_i = 0. \tag{21}$$

Here, the null vector field $k \in \mathfrak{X}$ is normalized ($k^0 = 1$) and $f$ is a smooth function. Exploiting the Kerr-Schild form, some curvature quantities take a particularly simple form, e.g.,

$$R = \partial_a \partial_b (f k^a k^b). \tag{22}$$

The Schwarzschild metric is a member of the Kerr-Schild class. In fact, transformation to Eddington-Finkelstein coordinates ($\bar{t} = t + 2m \log|2m - r|$) yields

$$ds^2 = -d\bar{t}^2 + dr^2 + r^2 d\Omega^2 + \frac{2m}{r}(d\bar{t} - dr)^2. \tag{23}$$

Evidently, (23) is of Kerr-Schild form, $g_{ij} = \eta_{ij} + f k_i k_j$ , with

$$k = \frac{\partial}{\partial \bar{t}} + \frac{\partial}{\partial r} \qquad \text{and} \qquad f = \frac{2m}{r}\,. \tag{24}$$

Analogously to section III, we regard $f$ and $k$ as distributions on $\mathbb{R}^4$. By this we again implicitly fix the differentiable structure: the coordinates $(\bar{t}, x^i)$ are extended through the origin.

We now proceed by regularizing *both* $f$ and $k$. Indeed, this is necessary due to the fact that not only the profile function $f$ is singular, but also the null vector field $k$ is non-smooth. Recall that, on account of the nonlinearities in $R[g]$, (22) can only be derived for smooth functions; it is inaccessible for distributional input. (Note the analogy of this situation with the one encountered in (8)). Hence, $f$ and $k$ are chosen to be the fundamental variables characterizing the metric (compare with the remarks in Section IV). Regularizing the function $f$ as in Section III gives

$$f(r) = \frac{2m}{r} \stackrel{\iota}{\to} \iota(f(r)) = f_\varepsilon(r) = 2m(\frac{1}{r})_\varepsilon\,. \tag{25}$$

The regularization of $k_i$ is carried out in detail in appendix C, yielding

$$k_i(r) = \frac{x_i}{r} \stackrel{\iota}{\longrightarrow} \iota(\frac{x_i}{r}) = k_{i\,\varepsilon} = (\frac{x_i}{r})_\varepsilon = x_i\, F_\varepsilon(r) \qquad (i = 1, \ldots, 3)\,, \tag{26}$$

where $F_\varepsilon$ is given by (C3). Note that, for the moment, $k_0 = 1$ is embedded trivially into $\mathcal{G}$. Collecting the results of (25) and (26) we get the regularized metric

$$ds_\varepsilon^2 = (-1 + f_\varepsilon)dt^2 - 2r f_\varepsilon F_\varepsilon\, dt dr + (1 + \tilde{f}_\varepsilon)dr^2 + r^2 d\Omega^2\,, \tag{27}$$

where, for simplicity, $\bar{t}$ has been replaced by $t$ again, and $\tilde{f}_\varepsilon$ abbreviates $r^2 f_\varepsilon F_\varepsilon^2$. Unfortunately, (27) is no longer of Kerr-Schild form. This can be shown indirectly: assuming that (27) is Kerr-Schild, i.e., assuming that $ds_\varepsilon^2 = -dt'^2 + dr^2 + r^2 d\Omega^2 + f'_\varepsilon\,(dt' - dr)^2$ can be achieved by a transformation $t \to t'(t, r)$, it follows that $f'_\varepsilon = \frac{\tilde{f}_\varepsilon}{1 - f_\varepsilon + \tilde{f}_\varepsilon}$ and $(\frac{dt}{dt'})^2 = \frac{1}{1 - f_\varepsilon + \tilde{f}_\varepsilon}$. As a consequence, the denominator $1 - f_\varepsilon + \tilde{f}_\varepsilon$ must be a strictly positive function. In fact, in the sense of association, it is equal to one. However, at the origin $r = 0$, we obtain $1 - f_\varepsilon + \tilde{f}_\varepsilon = 1 - 8\pi m \frac{1}{\varepsilon} \int_0^\infty t\rho(t)dt$ , which is negative for small $\varepsilon$ as long as $\int_0^\infty t\rho(t)dt > 0$, a contradiction.

The fact that the embedding (27) is no longer of Kerr-Schild form bears a strong relation to the fact that smooth regularizations degenerate at a certain value of $r$ (see Section IV): the determinant of (27) contains the factor $1 - f_\varepsilon + \tilde{f}_\varepsilon$ , which was shown to possess a zero.

Additionally, in analogy to the statements made at the end of Section IV, we note that the loss of the Kerr-Schild form does not stem from choosing to regularize the singular coefficient functions via convolution. On the contrary, it may be shown that any regularization of the metric displays this behavior as long as only the spatial components of $k$ are taken into account.

We will now take the announced geometrical view-point: we consider regularizations retaining the Kerr-Schild decomposition. This requires, in particular, that the regularized vector $k_\varepsilon$ is still null. Thus, we consider the regularization

$$k_\varepsilon^i = rF_\varepsilon\, k^i \qquad (i = 0\ldots 3)\,. \tag{28}$$

While the spatial components of (28) coincide with (26), $k_\varepsilon^0$ is only associated to 1 ($k_\varepsilon^0 = rF_\varepsilon \approx 1$). As required, $k_\varepsilon$ satisfies the condition $k_\varepsilon^i k_{\varepsilon\, i} = 0$. Note that, in order to obtain (28), the functions $f$, $k^i$ $(i = 1\ldots 3)$ and $k\cdot k$ are chosen as fundamental variables determining the geometric structure of the spacetime.

Using (25) and (28) the regularized metric takes the form

$$g_{ij\,\varepsilon} = \eta_{ij} + f_\varepsilon k_{\varepsilon\, i} k_{\varepsilon\, j} = \eta_{ij} + (r^2 F_\varepsilon^2 f_\varepsilon)\, k_i k_j = \eta_{ij} + \tilde{f}_\varepsilon\, k_i k_j. \tag{29}$$

Obviously, (29) is of Kerr-Schild form. Finally we have arrived at a regularization of the Schwarzschild metric which is *both smooth and invertible* (the inverse being $\eta^{ij} - \tilde{f}_\varepsilon k^i k^j$). This allows us to fully exploit the Kerr-Schild form, i.e., to use (22), to obtain

$$R_\varepsilon = \partial_a \partial_b (\tilde{f}_\varepsilon k^a k^b) = \frac{4}{r}\, \tilde{f}_\varepsilon' + \tilde{f}_\varepsilon'' + \frac{2}{r^2}\, \tilde{f}_\varepsilon\,. \tag{30}$$

To complete our program we calculate the weak limit of $R_\varepsilon$. The technically involved calculations are deferred to appendix D. Finally we derive

$$(R_\varepsilon)_\varepsilon \;\approx\; 8\pi m \delta\,. \tag{31}$$

The Ricci tensor can be treated in complete analogy to obtain the Einstein tensor

$$(G^a_{\;b\,\varepsilon})_\varepsilon \;\approx\; -8\pi m\, \delta\, \delta^a_0 \delta^0_b\,. \tag{32}$$

## VI. THE HORIZON

In this last section we leave the neighborhood of the singularity at the origin and turn to the singularity at the horizon. The question we are aiming at is the following: using distributional geometry (thus without leaving Schwarzschild coordinates), is it possible to show that the horizon singularity of the Schwarzschild metric is merely a coordinate singularity? In order to investigate this issue we calculate the distributional curvature at the horizon (in Schwarzschild coordinates).

Examining the Schwarzschild metric (6) in a neighborhood of the horizon, we see that, whereas $h(r)$ is smooth, $h^{-1}(r)$ is not even $L^1_{loc}$ (note that the origin is now always excluded from our considerations; the space we are working on is $\mathbb{R}^3\backslash\{0\}$). Thus, regularizing the Schwarzschild metric amounts to embedding $h^{-1}$ into $\mathcal{G}$ (as done in (5)).

$$ds_\varepsilon^2 = h(r)\, dt^2 - (h^{-1})_\varepsilon(r)\, dr^2 + r^2\, d\Omega^2 \tag{33}$$

$$\text{with} \quad h(r) = -1 + \frac{2m}{r} \quad \text{and} \quad (h^{-1})_\varepsilon(r) = -1 - 2m[\mathrm{vp}(\frac{1}{r-2m})]_\varepsilon \tag{34}$$

Obviously, (33) is degenerate at $r = 2m$, because $h(r)$ is zero at the horizon. However, this does not come as a surprise. Both $h(r)$ and $h^{-1}(r)$ are positive outside of the black hole and negative in the interior. As a consequence *any* (smooth) regularization of $h(r)$ (or $h^{-1}$) must pass through zero somewhere and, additionally, this zero must converge to $r = 2m$ as the regularization parameter goes to zero (note the analogy to the situation in section IV).

Due to the degeneracy of (33), the Levi-Cività connection is not available. Consider, therefore, the following connection $\Gamma^l_{kj} \in \mathcal{G}$:

$$\Gamma^l_{kj} = \frac{1}{2}[\iota(g^{-1})]^{\,lm}[\iota(g)_{mk,j} + \iota(g)_{mj,k} - \iota(g)_{kj,m}] \tag{35}$$

Clearly, $\Gamma$ coincides with the Levi-Cività connection on $\mathbb{R}^3\backslash\{r = 2m, r = 0\}$, as $\iota(g) = g$ and $\iota(g^{-1}) = g^{-1}$ there.

Unfortunately, $\Gamma$ does not respect the regularized metric $\iota(g)$ (33), i.e., $\iota(g)_{ij;k} \neq 0$, e.g., $\iota(g)_{00;1} = (1 - h(h^{-1})_\varepsilon)h'$. However, compatibility with the metric $\iota(g)$ is a priori ruled out by the following statement: there exists no connection whatsoever under which $\iota(g)$ would be a parallel tensor. To show this, just look at ($L^i{}_{jk}$ denoting a not necessarily torsion-free connection) $\iota(g)_{00;1} = \iota(g)_{00,1} - 2L^0{}_{10}\iota(g)_{00}$. At the horizon $\iota(g)_{00} = 0$, so that $\iota(g)_{00;1}|_{r=2m} = h'(2m) = -\frac{1}{2m} \neq 0$. In the sense of association, however, the connection (35) is in fact metric compatible: $\iota(g)_{ij;k} \approx 0$.

We now investigate the curvature pertaining to the connection (35), picking out $R_{00\,\varepsilon}$ as a characteristic example. The result of the calculations displays the following structure

$$R_{00\,\varepsilon}(r) = \mathrm{v}p'_\varepsilon(-\frac{m^2}{r^2} + 4\frac{m^3}{r^3} - 4\frac{m^4}{r^4}) + \mathrm{v}p_\varepsilon(2\frac{m^3}{r^4} - 4\frac{m^4}{r^5}) + (-\frac{m^2}{r^4} - 2\frac{m^3}{r^5}) \tag{36}$$

$$= \mathrm{v}p'_\varepsilon(r)\sum_{l=2}^{\infty} c_l x^l + \mathrm{v}p_\varepsilon(r)\sum_{l=1}^{\infty} d_l x^l - \frac{1}{8m^2} - \frac{1}{16m^2}\sum_{l=1}^{\infty} e_l x^l \tag{37}$$

Here, the abbreviations $\mathrm{v}p_\varepsilon = [\mathrm{v}p(\frac{1}{r-2m})]_\varepsilon$ and $x = \frac{r-2m}{2m}$ are used; $c_l$, $d_l$ and $e_l$ are constants. Equation (37) holds for $|x| < 1$; the infinite sums converge in this case.

If the horizon is excluded, $R_{00\,\varepsilon} = 0$ (mod $\mathcal{N}$), because (35) coincides with the Schwarzschild Levi-Cività connection there. In the neighborhood of $r = 2m$ we aim at comparing $R_{00\,\varepsilon}(r)$ with a Colombeau object of the type $f(\frac{r-2m}{\varepsilon})$ ($f$ a Schwartz function). To this end we choose a fundamental sequence $r_\varepsilon = 2m + \varepsilon^q a_0$ converging to $r = 2m$ and examine $R_{00\,\varepsilon}(r_\varepsilon)$ (use (37) together with (5)).

- $q > 1$: $R_{00\,\varepsilon}(r_\varepsilon) = \mathrm{const} + o(\varepsilon^{q-1})$.
- $q < 1$: Using (5) we find that $\mathrm{v}p_\varepsilon(r_\varepsilon) = \frac{1}{r_\varepsilon - 2m}$ and $\mathrm{v}p'_\varepsilon(r_\varepsilon) = -\frac{1}{(r_\varepsilon-2m)^2}$ (in the sense of generalized numbers). Inserting these results into equation (36), we obtain $R_{00\,\varepsilon}(r_\varepsilon) = 0$.
- $q = 1$: $R_{00\,\varepsilon}(r_\varepsilon) = \mathrm{const} + o(1)$.

Thus, $R_{00\,\varepsilon}(r_\varepsilon)$ has the same asymptotic behavior as a sequence of the type $f(\frac{r_\varepsilon - 2m}{\varepsilon})$ (as $\varepsilon \to 0$). As a consequence, the weak limit of $R_{00\varepsilon}(r)$ can be calculated easily, simply by evaluating $\int dr\, r^2\tilde{\Phi}(r) f(\frac{r-2m}{\varepsilon})$. Evidently, this expression vanishes as $\epsilon \to 0$. Since analogous results hold for the other components of the Ricci tensor, we are finally able to state

$$R_{ij\,\varepsilon} \approx 0\,. \tag{38}$$

In other words: viewed as a distribution, $R_{ij} = 0$ on $\mathbb{R}^3 \backslash \{0\}$, i.e., including the horizon. If we were courageous enough we could take this as a proof that the metric singularity at the horizon is only a coordinate singularity.

We conclude this section with a remark on the connection (35). Due to the degeneracy of any regularization of the metric (e.g. (33)) no canonical (Levi-Cività) connection could be defined. The choice of connection (35) bears a strong relation to the regularized metric; however, there seems no way of telling if this choice is canonical in some sense and thus preferred to other choices. Despite this open question, at least it is clear that the connection (35) is a regularization of the Schwarzschild *connection*. Indeed, we could change our viewpoint: we consider the Schwarzschild connection (forgetting where it came from, i.e., forgetting about the metric), regularize its distribution-valued components and calculate the distributional curvature from it. Proceeding in this manner, we obtain the result (38), i.e., the spacetime is weakly Ricci-flat (the origin was excluded from our considerations).

## APPENDIX A: MOLLIFIER INTEGRALS

Throughout this paper we work invariably with radially symmetric mollifiers $\rho(r)$ (cf. section II). Most importantly, we have the properties

$$\int_0^\infty dt\, t^2 \rho(t) = \frac{1}{4\pi} \qquad\qquad \int_0^\infty dt\, t^{2k}\rho(t) = 0 \ \ (k > 1)\,. \tag{A1}$$

We investigate multiple integrals involving the mollifier $\rho(r)$ and powers of $r$:

$$\int_x^\infty dt\, t^n \int_t^\infty s\rho(s)\, ds = -\frac{x^{n+1}}{n+1}\int_x^\infty t\rho(t) dt + \frac{1}{n+1}\int_x^\infty dt\, t^{n+2}\rho(t) \tag{A2}$$

(A2) holds for $(n, k \neq -1)$, it is proven by simply performing integration by parts.

One of the most interesting cases resulting from (A2) is $n = 0$ and $x \to 0$:

$$\int_0^\infty dt \int_t^\infty s\rho(s)ds = \frac{1}{4\pi} \tag{A3}$$

## APPENDIX B: EMBEDDING OF THE CARTESIAN COMPONENTS

Referring to section IV we investigate

$$\iota\left(\frac{1 + h(r)^{-1}}{r^2} x_i x_j\right) dx^i dx^j \; = \; \left(2m \int \mathrm{f}(\|\vec{z}\|)\, z_i z_j\, \rho_\varepsilon(\|\vec{z} + \vec{x}\|)\, d^3z\right) dx^i dx^j \,, \tag{B1}$$

where $f(q) = \frac{1}{2m-q}\,\frac{1}{q^2}$. In order to simplify (B1) we show the following relation

$$\iota\left(\frac{1 + h(r)^{-1}}{r^2} x_i x_j\right) = x_i\, x_j\, c_\varepsilon(\vec{x}) \quad \text{for } i \neq j \qquad (c_\varepsilon \text{ smooth}). \tag{B2}$$

Proof: Since both $f(\|\vec{z}\|)$ and $\rho_\varepsilon$ are even functions in $z_i$, we observe that

$$\iota(\tfrac{1+h(r)^{-1}}{r^2} x_i x_j)\Big|_{x_i = 0} = 2m \int f(\|\vec{z}\|) z_i z_j \rho_\varepsilon(\ldots, z_i, \ldots) d^3z = 0\,. \tag{B3}$$

We can conclude that

$$\iota(\tfrac{1+h(r)^{-1}}{r^2} x_i x_j) = x_i\, c'_\varepsilon(\vec{x}) \qquad (c'_\varepsilon \text{ smooth}) \tag{B4}$$

Also, $\iota(\frac{1+h(r)^{-1}}{r^2} x_i x_j)|_{x_j=0} = 0$, from which follows that $c'_\varepsilon(\vec{x}) = x_j c_\varepsilon(\vec{x})$, yielding (B2). Note, however, that the smooth function $c_\varepsilon(\vec{x})$ in (B2) is not equal to $\iota(\frac{1+h(r)^{-1}}{r^2})$. ∎

In the case $i = j$, equation (B2) is no longer valid. We are able to calculate the $ii-$component in the limiting case $\varepsilon \to 0$, i.e.,

$$\iota\left(\frac{1 + h(r)^{-1}}{r^2} x_i^2\right)\Big|_{\vec{x}=0} = 2m \int f(\|\vec{z}\|) z_i^2 \rho_\varepsilon(\|\vec{z}\|) d^3z \;\to\; \frac{1}{3} \quad (\varepsilon \to 0). \tag{B5}$$

Proof: Clearly, $2m \int f(\|\vec{z}\|) z_i^2 \rho_\varepsilon(\|\vec{z}\|) d^3z$ is independent of the choice of the index $i$, so that we may substitute it by $\frac{2m}{3} \int f(\|\vec{z}\|) \|\vec{z}\|^2 \rho_\varepsilon(\|\vec{z}\|) d^3z$. Obviously, this converges to $\frac{1}{3}$ as $\varepsilon$ goes to zero. ∎

The regularized metric (B1) is radially symmetric, $R^* g_\varepsilon = g_\varepsilon$ ($R$ a rotation), as long as radially symmetric mollifiers are used. Thus, it must be of the form of a general radially symmetric metric $ds^2 = a(r)\, dr^2 + r^2 b(r)\, d\Omega^2$, hence

$$\iota\left(\frac{1 + h(r)^{-1}}{r^2} x_i x_j\right) dx^i dx^j = (a_\varepsilon - b_\varepsilon)(r) \frac{x_i}{r} \frac{x_j}{r} dx^i dx^j + b_\varepsilon(r)\, d\vec{x}^2\,. \tag{B6}$$

For the general radially symmetric metric (B6) to be smooth $(a_\varepsilon - b_\varepsilon)(r) = O(r^2)$ is required. We observe consistency with (B2) and conclude

$$a_\varepsilon(r) = b_\varepsilon(r) + c_\varepsilon r^2\,. \tag{B7}$$

At the origin $r = 0$ only the second term $b(r)\, d\vec{x}^2$ remains relevant, since $(a - b)(r) \frac{x_i}{r} \frac{x_j}{r}|_{r=0} = c_\varepsilon x_i x_j|_{r=0} = 0$. Turning equation (B5) to good account we obtain

$$b_\varepsilon(0) \to \frac{1}{3} \quad (\varepsilon \to 0), \qquad a_\varepsilon(0) \to \frac{1}{3} \quad (\varepsilon \to 0)\,. \tag{B8}$$

Combining (17) with (B6), we finally obtain

$$ds_\varepsilon^2 = h_\varepsilon(r)\, dt^2 + (1 - a_\varepsilon(r)) dr^2 + (1 - b_\varepsilon(r)) r^2 d\Omega^2\,. \tag{B9}$$

## APPENDIX C: EMBEDDING OF $k^i$

We explicitly embed the radially outward pointing unit vector field $k_i = \frac{x_i}{r}$ $(i = 1 \cdots 3)$ into the Colombeau algebra, i.e.,

$$\iota(\frac{x_i}{r}) = \int d^3x' \, \frac{x_i - x_i'}{\|\vec{x} - \vec{x}'\|} \, \rho_\varepsilon(\|\vec{x}'\|) = -\int d^3z \, \frac{z_i}{\|\vec{z}\|} \, \rho_\varepsilon(\|\vec{x} + \vec{z}\|) \,. \tag{C1}$$

Equation (C1) is of an analogous form as (B1) in appendix B. We may conclude that $\iota(\frac{x_i}{r})$ is a radially symmetric vector field. Moreover, despite $\iota(\frac{x_i}{r}) \neq x_i \iota(\frac{1}{r})$ , we must still have (repeating (B2)ff.)

$$\iota(\frac{x_i}{r}) = x_i \, F_\varepsilon(\vec{x}) \qquad (i = 1 \ldots 3) \,. \tag{C2}$$

Here, $F_\varepsilon(\vec{x})$ is a smooth function and moreover, because this function must be radially symmetric, $F_\varepsilon(\vec{x}) = F_\varepsilon(r)$.

This fact makes it possible to calculate $\iota(\frac{x_i}{r})$ explicitly. Take $x = (0, 0, r)$ and investigate $\iota(\frac{x_3}{r}) = x_3 F_\varepsilon(r) = r F_\varepsilon(r)$:

$$\begin{aligned} rF_\varepsilon(r) &= \int \frac{x_3 - x_3'}{\|\vec{x} - \vec{x}'\|} \, \rho_\varepsilon(r') \, d^3x' \\ &= 2\pi \int r'^{\,2} dr' \, \rho_\varepsilon(r') \int_{-1}^{1} d(\cos\theta') \, \frac{r - r'\cos\theta'}{\sqrt{r^2 + r'^{\,2} - 2rr'\cos\theta'}} \\ &= r(\frac{1}{r})_\varepsilon - 2\pi \int r'^{\,2} dr' \, \rho_\varepsilon \left( \frac{1}{r}(|r - r'| + (r + r')) + \frac{1}{3}\frac{1}{r^2}\frac{1}{r'}(|r - r'|^3 - (r + r')^3) \right) \\ &= r \left( \frac{4\pi}{r} \int_0^r ds \, s^2 \rho_\varepsilon(s) + \frac{8\pi}{3} \int_r^\infty ds \, s \rho_\varepsilon(s) - \frac{4\pi}{3}\frac{1}{r^3} \int_0^r ds \, s^4 \rho_\varepsilon(s) \right) \end{aligned} \tag{C3}$$

Clearly, $F_\varepsilon(r) = \frac{1}{r}$ on $\mathbb{R}^3 \backslash \{0\}$ and, moreover, $F_\varepsilon(r) \approx \frac{1}{r}$ on the whole space. We can write the latter also in the form $rF_\varepsilon(r) \approx 1$.

## APPENDIX D: WEAK LIMITS FOR THE KERR-SCHILD CASE

We investigate the distributional limit of (30). Inserting for $\tilde{f}_\varepsilon$, (30) becomes

$$\begin{aligned} R_\varepsilon \;=\; \frac{128\pi^3 m}{3} \Big( & 16[1]_\varepsilon^3 + \frac{4}{r^7}[2]_\varepsilon[4]_\varepsilon^2 + \frac{2}{r^6}[1]_\varepsilon[4]_\varepsilon^2 - \frac{4}{r^5}[4]_\varepsilon[2]_\varepsilon^2 + \\ & + \frac{32}{r}[2]_\varepsilon[1]_\varepsilon^2 + \frac{14}{r^2}[1]_\varepsilon[2]_\varepsilon^2 - 3\rho_\varepsilon[2]_\varepsilon^2 - 4r\rho_\varepsilon[1]_\varepsilon[2]_\varepsilon + \\ & - \frac{4r^2}{3}\rho_\varepsilon[1]_\varepsilon^2 - \frac{1}{3r^4}\rho_\varepsilon[4]_\varepsilon^2 + 2\frac{1}{r^2}\rho_\varepsilon[2]_\varepsilon[4]_\varepsilon + \frac{4}{3r}\rho_\varepsilon[1]_\varepsilon[4]_\varepsilon \Big) \,, \end{aligned} \tag{D1}$$

where $[1]_\varepsilon := \int_r^\infty ds \, s\rho_\varepsilon(s)$, $[2]_\varepsilon := \int_0^r ds \, s^2\rho_\varepsilon(s)$, $[4]_\varepsilon := \int_0^r ds \, s^4\rho_\varepsilon(s)$.

In order to compute the weak limit of (D1), expressions of the form (D2) and (D3) below have to be investigated. (Note that the negative powers of $r$ are compensated by the integrals so that both (D2) and (D3) are well-defined as $r \to 0$).

$$r^{-4+2j+3i} \, \rho_\varepsilon(r) \;\; [1]_\varepsilon^{\,i} \, [2]_\varepsilon^{\,j} \, [4]_\varepsilon^{\,k} \qquad (i + j + k = 2) \tag{D2}$$

$$r^{-9+2j+3i} \;\; [1]_\varepsilon^{\,i} \, [2]_\varepsilon^{\,j} \, [4]_\varepsilon^{\,k} \qquad (i + j + k = 3) \tag{D3}$$

Terms of the forms (D2) and (D3) possess related distributional limits:

$$\begin{aligned} \left(r^{-9+2j+3i} \, [1]_\varepsilon^{\,i} \, [2]_\varepsilon^{\,j} \, [4]_\varepsilon^{\,k}\right)_\varepsilon \;\approx\; & +\frac{i}{2j + 3i - 6} \left(r^{-7+2j+3i}\rho_\varepsilon(r)[1]_\varepsilon^{\,i-1} \, [2]_\varepsilon^{\,j} \, [4]_\varepsilon^{\,k}\right)_\varepsilon + \\ & -\frac{j}{2j + 3i - 6} \left(r^{-6+2j+3i}\rho_\varepsilon(r)[1]_\varepsilon^{\,i} \, [2]_\varepsilon^{\,j-1} \, [4]_\varepsilon^{\,k}\right)_\varepsilon + \\ & -\frac{k}{2j + 3i - 6} \left(r^{-4+2j+3i}\rho_\varepsilon(r)[1]_\varepsilon^{\,i} \, [2]_\varepsilon^{\,j} \, [4]_\varepsilon^{\,k-1}\right)_\varepsilon \,. \end{aligned}$$

In order to show this, consider the distributional action on a test function $\Phi(\vec{x})$, i.e., $\int_0^\infty dr\, \tilde{\Phi}(r) r^{-7+2j+3i}\, [1]_\varepsilon^{\,i}\, [2]_\varepsilon^{\,j}\, [4]_\varepsilon^{\,k}$. Here, we have introduced $\tilde{\Phi}(r) := \int \sin\theta d\theta d\phi\, \Phi(\vec{x})$. Integrating by parts and using $\tilde{\Phi}'(0) = 0$, the claim is established.

Taking $(j = 2; k = 1)$ as an example, we obtain the following weak limit:

$$
\begin{aligned}
\int dr\, \tilde{\Phi}(r) \frac{1}{r^3}\, [2]_\varepsilon^2 [4]_\varepsilon \quad &\stackrel{\varepsilon\to 0}{\longrightarrow} \quad \tilde{\Phi}(0) \int dx \rho(x) [2][4] + \frac{1}{2} \tilde{\Phi}(0) \int dx\, x^2 \rho(x) [2]^2 \\
&\stackrel{(A1)}{=} \quad \tilde{\Phi}(0) \int dx \rho(x) [2][4] + \frac{1}{6} \frac{1}{64\pi^3} \tilde{\Phi}(0)\, .
\end{aligned}
$$

Here, $\qquad [1] := \int_x^\infty dt\, t \rho(t) \qquad [2] := \int_0^x dt\, t^2 \rho(t) \qquad [4] := \int_0^x dt\, t^4 \rho(t)\, .$

Eventually, we obtain the result (31) for the distributional limit of (D1).

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